

Robust Estimation of Optical Phase Varying as a Continuous Resonant Process

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Abstract—It is well-known that adaptive homodyne estimation of continuously varying optical phase provides superior accuracy in the phase estimate as compared to adaptive or non-adaptive static estimation. However, most phase estimation schemes rely on precise knowledge of the underlying parameters of the system under measurement, and performance deteriorates significantly with changes in these parameters; hence it is desired to develop robust estimation techniques immune to such uncertainties. In related works, we have already shown how adaptive homodyne estimation can be made robust to uncertainty in an underlying parameter of the phase varying as a simplistic Ornstein-Uhlenbeck stochastic noise process. Here, we demonstrate robust phase estimation for a more complicated resonant noise process using a guaranteed cost robust filter.

I. INTRODUCTION

Quantum phase estimation is the problem of estimating an unknown classical phase involved in the dynamics of a quantum system [1]. Precise phase estimation plays a key role in quantum computation [2], communication [3], [4] and metrology [5]. A fundamental bound on precision is imposed by quantum mechanics [6] and this limits gravitational wave detection [7] and can guarantee security in quantum cryptography [8].

Adaptive homodyne *single-shot* measurements of a *fixed* unknown phase can yield mean-square estimation errors below the standard quantum limit (SQL) [9], [10], [11], [12], [13], which is the minimum level of quantum noise obtained using standard measurements without real-time feedback. It is, however, practically more desirable to precisely track a *continuously varying* phase [14], [15], [16], [17], [18].

In Refs. [17], [18] the signal phase to be estimated is allowed to continuously evolve under the influence of an unmeasured classical stochastic Ornstein-Uhlenbeck (OU) noise process. However, since it is physically unreasonable to precisely know the underlying parameters of the noise process, the estimation process is significantly affected due to the uncertainty in these parameters. Hence, it is desired to make the phase estimation robust to uncertainties in these parameters.

The authors have previously demonstrated in Ref. [19] that using a robust feedback filter designed based on a guaranteed cost robust filtering approach [20] can improve the estimation process as compared to an optimal filter. In related works, the authors have also shown how continuous phase estimation using smoothing (rather than filtering alone)

can be made robust to uncertainties in the phase being measured and/or the measured output for a coherent beam of light [21] and a phase-squeezed beam of light [22], by employing robust fixed-interval smoothing theory from Ref. [23] for continuous uncertain systems admitting a certain integral quadratic constraint.

These works, however, dealt with estimating the signal phase modulated by an Ornstein-Uhlenbeck noise process, which turns out to be a much simplified noise model as compared to the kind of noises that in practice corrupt the signal phase to be estimated. We, therefore, consider a more relevant and complicated second-order resonant noise process here and design a guaranteed cost robust feedback filter based on Ref. [20] for adaptive homodyne phase estimation of a coherent light beam phase-modulated by the same.

II. RESONANT NOISE PROCESS

The resonant noise process we consider here is the one typically generated by a piezo-electric transducer (PZT) driven by an input white noise as in Fig. 1.

A. Transfer Function

The simplified transfer function of a typical PZT is given by:

$$G(s) := \frac{\phi}{v} = \frac{\kappa}{s^2 + 2\zeta\omega_r s + \omega_r^2}, \quad (1)$$

where κ is the gain, ζ is the damping factor, ω_r is the resonant frequency (rad/s) and v is a zero-mean white Gaussian noise with unity amplitude.

We use the following values for the parameters above: $\kappa = 1$, $\zeta = 0.01$ and $\omega_r = 6.283 \times 10^3$ rad/s, i.e. 1 kHz. The corresponding Bode plot of the transfer function (1) is given in Fig. 2.

B. State-Space Realization

From (1), we obtain:

$$\ddot{\phi}(t) + 2\zeta\omega_r\dot{\phi}(t) + \omega_r^2\phi(t) = \kappa v(t). \quad (2)$$

Let $x_1 := \phi$ and $x_2 := \dot{\phi}$. Then, we have:

$$\dot{x}_1 = x_2, \quad (3)$$

$$\dot{x}_2 = -2\zeta\omega_r x_2 - \omega_r^2 x_1 + \kappa v. \quad (4)$$

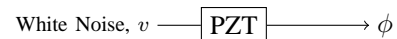


Fig. 1. Resonant noise

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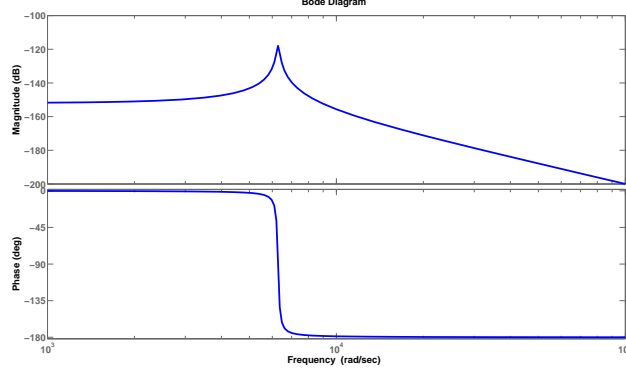


Fig. 2. Bode plot of the resonant noise process.

Let $\mathbf{x} := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

So, the state-space realization of the PZT output is:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{G}v, \quad (5)$$

where

$$\mathbf{A} := \begin{bmatrix} 0 & 1 \\ -\omega_r^2 & -2\zeta\omega_r \end{bmatrix}, \quad \mathbf{G} := \begin{bmatrix} 0 \\ \kappa \end{bmatrix}.$$

III. OPTIMAL KALMAN FILTER

The standard adaptive homodyne phase estimation of a coherent state of light is optimal for given values of the parameters when using a Kalman filter in the feedback loop to adapt the phase of the local oscillator. Under a linearization approximation, the homodyne photocurrent from the adaptive phase estimation system is given by [17]:

$$I(t)dt = 2|\alpha|[\phi(t) - \hat{\phi}(t)]dt + dW(t), \quad (6)$$

where $|\alpha|$ is the amplitude of the coherent state with photon flux given by $\mathcal{N} := |\alpha|^2$, $\hat{\phi}$ is the *intermediate phase estimate*, and dW is Wiener noise arising from quantum vacuum fluctuations.

The *instantaneous estimate* $\theta(t)$ is given by [17]:

$$\theta(t) := \hat{\phi}(t) + \frac{I(t)}{2|\alpha|} = \phi(t) + \frac{1}{2|\alpha|}w(t), \quad (7)$$

where $w := \frac{dW}{dt}$ is a zero-mean Gaussian white noise with unity amplitude.

Expressing (7) in terms of \mathbf{x} defined in the previous section, we get the measurement model as:

$$\theta = \mathbf{H}\mathbf{x} + \mathbf{J}w, \quad (8)$$

where $\mathbf{H} := \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\mathbf{J} := \begin{bmatrix} \frac{1}{2|\alpha|} \end{bmatrix}$.

Rewriting the equations for the system under consideration, we get:

$$\begin{aligned} \text{Process model: } \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{G}v, \\ \text{Measurement model: } \theta &= \mathbf{H}\mathbf{x} + \mathbf{J}w, \end{aligned} \quad (9)$$

where

$$\begin{aligned} E[v(t)v(\tau)] &= \mathbf{R}\delta(t - \tau), \\ E[w(t)w(\tau)] &= \mathbf{S}\delta(t - \tau), \\ E[v(t)w(\tau)] &= 0. \end{aligned}$$

Since v and w are unity amplitude white noise processes, both \mathbf{R} and \mathbf{S} are unity (scalars).

The continuous-time algebraic Riccati equation to be solved to construct the steady-state Kalman filter for the system is then [24]:

$$\mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T + \mathbf{G}\mathbf{R}\mathbf{G}^T - \mathbf{P}\mathbf{H}^T(\mathbf{J}\mathbf{S}\mathbf{J}^T)^{-1}\mathbf{H}\mathbf{P} = \mathbf{0}. \quad (10)$$

The Kalman filter equation is [24]:

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}\mathbf{H})\hat{\mathbf{x}} + \mathbf{K}\mathbf{H}\mathbf{x} + \mathbf{K}\mathbf{J}w, \quad (11)$$

where $\mathbf{K} := \mathbf{P}\mathbf{H}^T(\mathbf{J}\mathbf{S}\mathbf{J}^T)^{-1}$ is the Kalman gain.

Using $\kappa = 1$, $\zeta = 0.01$ and $\omega_r = 6.283 \times 10^3$ rad/s, i.e. 1 kHz as in section II-A and $|\alpha| = 6 \times 10^8$, we get:

$$\mathbf{P} = \begin{bmatrix} 3.33785970 \times 10^{-14} & 8.02174133 \times 10^{-10} \\ 8.02174133 \times 10^{-10} & 3.99751822 \times 10^{-5} \end{bmatrix}, \quad (12)$$

and

$$\mathbf{K} = \begin{bmatrix} 4.80651797 \times 10^4 \\ 1.15513075 \times 10^9 \end{bmatrix}. \quad (13)$$

IV. ROBUST FILTER

In this section, we make our filter robust to uncertainties in the parameters underlying the system matrix \mathbf{A} using the guaranteed cost estimation robust filtering approach given in Ref. [20].

We introduce uncertainty in \mathbf{A} as follows:

$$\mathbf{A} \rightarrow \mathbf{A} + \begin{bmatrix} 0 & 0 \\ -\mu_1\delta_1\omega_r^2 & -2\mu_2\delta_2\zeta\omega_r \end{bmatrix},$$

where $\Delta := \begin{bmatrix} \delta_1 & \delta_2 \end{bmatrix}$ is an uncertain parameter satisfying $\|\Delta\| \leq 1$, i.e. $\delta_1^2 + \delta_2^2 \leq 1$, and $0 \leq \mu_1 < 1$, $0 \leq \mu_2 < 1$ are parameters determining the levels of uncertainty.

The process and measurement models from (9) now take the form:

$$\begin{aligned} \text{Process model: } \dot{\mathbf{x}} &= (\mathbf{A} + \mathbf{D}_1\Delta\mathbf{E}_1)\mathbf{x} + \mathbf{G}v, \\ \text{Measurement model: } \theta &= \mathbf{H}\mathbf{x} + \mathbf{J}w, \end{aligned} \quad (14)$$

where

$$\mathbf{D}_1 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{E}_1 := \begin{bmatrix} -\mu_1\omega_r^2 & 0 \\ 0 & -2\mu_2\zeta\omega_r \end{bmatrix}.$$

As in Ref. [20], the Riccati equation to be solved in order to construct the guaranteed cost filter for the system is:

$$\begin{aligned} \mathbf{A}\mathbf{Q} + \mathbf{Q}\mathbf{A}^T + \epsilon\mathbf{Q}\mathbf{E}_1^T\mathbf{E}_1\mathbf{Q} - \epsilon\mathbf{Q}\mathbf{H}^T(\epsilon\mathbf{J}\mathbf{S}\mathbf{J}^T)^{-1}\mathbf{H}\mathbf{Q} \\ + \frac{1}{\epsilon}\mathbf{D}_1\mathbf{D}_1^T + \mathbf{G}\mathbf{R}\mathbf{G}^T = \mathbf{0}. \end{aligned} \quad (15)$$

The stabilising solution of this equation yields an upper bound $\tilde{\mathbf{Q}}$ for the robust filter error covariance. It is desired to obtain the optimum value of ϵ (that we call ϵ_{opt}) at which the element in the first row and first column of the matrix

$\tilde{\mathbf{Q}}$ (that we denote by $Q^+ := \tilde{\mathbf{Q}}(1,1)$) is minimum. For example, for $\mu_1 = 0.5, \mu_2 = 0$ and nominal values for other parameters, Fig. 3 shows the plot of Q^+ versus ϵ , where ϵ_{opt} is found to be 35. Thus, we get:

$$\tilde{\mathbf{Q}} = \begin{bmatrix} 3.38608462 \times 10^{-14} & 8.17703018 \times 10^{-10} \\ 8.17703018 \times 10^{-10} & 4.09328251 \times 10^{-5} \end{bmatrix}. \quad (16)$$

The robust filter equation is:

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} + \epsilon \tilde{\mathbf{Q}} \mathbf{E}_1^T \mathbf{E}_1 - \tilde{\mathbf{Q}} \mathbf{H}^T (\mathbf{J} \mathbf{S} \mathbf{J}^T)^{-1} \mathbf{H}) \hat{\mathbf{x}} + \tilde{\mathbf{Q}} \mathbf{H}^T (\mathbf{J} \mathbf{S} \mathbf{J}^T)^{-1} \mathbf{H} \mathbf{x} + \tilde{\mathbf{Q}} \mathbf{H}^T (\mathbf{J} \mathbf{S} \mathbf{J}^T)^{-1} \mathbf{J} w. \quad (17)$$

V. COMPARISON OF THE ROBUST FILTER WITH THE KALMAN FILTER

A. Lyapunov Method

We augment the system given by (14) with the feedback filter (17) and represent the augmented system by the state-space model:

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}} \bar{\mathbf{x}} + \bar{\mathbf{B}} \bar{\mathbf{w}}, \quad (18)$$

where

$$\bar{\mathbf{x}} := \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{w}} := \begin{bmatrix} v \\ w \end{bmatrix}.$$

Thus, we have:

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} + \mathbf{D}_1 \Delta \mathbf{E}_1 & \mathbf{0} \\ \mathbf{F} & \mathbf{L} \end{bmatrix},$$

where $\mathbf{F} := \tilde{\mathbf{Q}} \mathbf{H}^T (\mathbf{J} \mathbf{S} \mathbf{J}^T)^{-1} \mathbf{H}$, $\mathbf{L} := \mathbf{A} + \epsilon \tilde{\mathbf{Q}} \mathbf{E}_1^T \mathbf{E}_1 - \tilde{\mathbf{Q}} \mathbf{H}^T (\mathbf{J} \mathbf{S} \mathbf{J}^T)^{-1} \mathbf{H}$ and

$$\bar{\mathbf{B}} = \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{Q}} \mathbf{H}^T (\mathbf{J} \mathbf{S} \mathbf{J}^T)^{-1} \mathbf{J} \end{bmatrix}.$$

For the continuous-time state-space model (18), the steady-state state covariance matrix \mathbf{P}_S is obtained by solving the *Lyapunov equation*:

$$\bar{\mathbf{A}} \mathbf{P}_S + \mathbf{P}_S \bar{\mathbf{A}}^T + \bar{\mathbf{B}} \bar{\mathbf{B}}^T = \mathbf{0}, \quad (19)$$

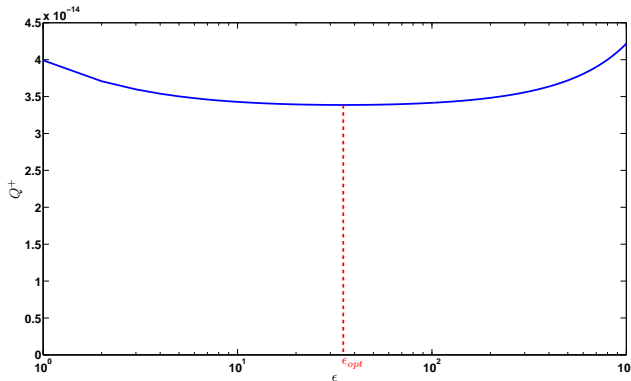


Fig. 3. Q^+ versus ϵ for $\mu_1 = 0.5$ and $\mu_2 = 0$.

where \mathbf{P}_S is the symmetric matrix

$$\mathbf{P}_S := E(\bar{\mathbf{x}} \bar{\mathbf{x}}^T) := \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{P}_2^T & \mathbf{P}_3 \end{bmatrix}.$$

The state estimation error can be written as:

$$\mathbf{e} := \mathbf{x} - \hat{\mathbf{x}} = [\mathbf{1} \quad -\mathbf{1}] \bar{\mathbf{x}},$$

which is mean zero since all of the quantities determining \mathbf{e} are mean zero.

The error covariance matrix is then given as:

$$\begin{aligned} \Sigma := E(\mathbf{e} \mathbf{e}^T) &= [\mathbf{1} \quad -\mathbf{1}] E(\bar{\mathbf{x}} \bar{\mathbf{x}}^T) \begin{bmatrix} \mathbf{1} \\ -\mathbf{1} \end{bmatrix} \\ &= [\mathbf{1} \quad -\mathbf{1}] \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_2 \\ \mathbf{P}_2^T & \mathbf{P}_3 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ -\mathbf{1} \end{bmatrix} = \mathbf{P}_1 - \mathbf{P}_2 - \mathbf{P}_2^T + \mathbf{P}_3. \end{aligned}$$

Since we are mainly interested in estimating $x_1 = \phi$, the estimation error covariance of interest is $\sigma^2 = \Sigma(1,1)$.

B. Standard Quantum Limit

The standard quantum limit is set by the minimum error in phase estimation that can be obtained using perfect heterodyne scheme. The SQL for our resonant noise process may be obtained using the same technique employed in Ref. [21] in the case of OU noise process.

We use the fact that the heterodyne scheme of measurement is, in principle, equivalent to, and incurs the same noise penalty as, *dual-homodyne* scheme [17], such as the schematic depicted in Fig. 4. A signal at the input is phase-modulated using an electro-optic modulator (EOM) that is driven by the resonant noise source. The modulated signal is then split using a 50 – 50 beamsplitter into two arms each with a homodyne detector (HD1 and HD2, respectively, with the local oscillator phase of HD1 $\pi/2$ out of phase with that of HD2). The ratio of the output signals of the two arms goes to an *arctan* block, the output of which is fed into a low-pass filter (LPF). The phase estimation error would be minimum when this LPF is an optimal Kalman filter.

The output signals of the two arms are:

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{2}} (2|\alpha| \sin \phi + n_1 + n_2), \\ I_2 &= \frac{1}{\sqrt{2}} (2|\alpha| \cos \phi + n_3 - n_4), \end{aligned}$$

where n_1 and n_3 are measurement noises of the two homodyne detectors, respectively, and n_2 and n_4 are the noises arising from the vacuum entering the empty port of the input beamsplitter corresponding to the two arms, respectively. All these noises are assumed to be zero-mean white Gaussian.

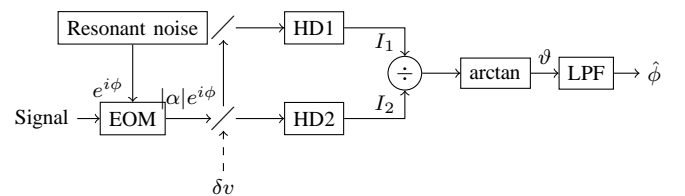


Fig. 4. Block diagram of the dual-homodyne scheme for deducing the SQL for the resonant noise.

The output of the arctan block is:

$$\vartheta = \arctan \left(\frac{2|\alpha| \sin \phi + n_1 + n_2}{2|\alpha| \cos \phi + n_3 - n_4} \right). \quad (20)$$

Assuming the input noises are small, a Taylor series expansion upto first-order terms of the right-hand side yields:

$$\vartheta \approx \phi + \frac{1}{2|\alpha|} n_1 + \frac{1}{2|\alpha|} n_2. \quad (21)$$

Expressing this equation in terms of \mathbf{x} , we get the measurement model as:

$$\boxed{\vartheta = \mathbf{H}\mathbf{x} + \mathbf{J}w,} \quad (22)$$

where $\mathbf{H} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\mathbf{J} = \begin{bmatrix} \frac{1}{2|\alpha|} & \frac{1}{2|\alpha|} \end{bmatrix}$.

The error covariance matrix of the optimal steady-state Kalman filter for the process given by (5) and the measurement given by (22) may be obtained by solving an algebraic Riccati equation of the form (10) for \mathbf{P} . The error covariance of interest (i.e. that in estimating $x_1 = \phi$) is then $\sigma^2 = \mathbf{P}(1, 1)$.

C. Comparison of Estimation Errors

The estimation errors may be calculated, as described in section V-A, for the robust filter, and likewise for the Kalman filter, as a function of δ_1 with $\delta_2 = 0$ for the nominal values of the parameters and chosen values for μ_1 with $\mu_2 = 0$. These were used to generate plots of the errors versus δ_1 to be able to compare the performance of the robust filter vis-a-vis the Kalman filter for the uncertain system with respect to the SQL. The SQL value is obtained by designing, as described in section V-B, a different Kalman filter for each value of the uncertain parameter. Figs. 5, 6 and 7 show these plots for $\mu_1 = 0.2, 0.5$ and 0.8 , respectively.

Clearly, as δ_1 deviates away from 0 towards -1 , the performance of the robust filter becomes superior than that of the Kalman filter in relation to the SQL for all levels of μ_1 .

Similarly, the estimation errors may be calculated, and plots created, for the robust filter and the Kalman filter as a function of δ_2 with $\delta_1 = 0$ for the nominal values of the

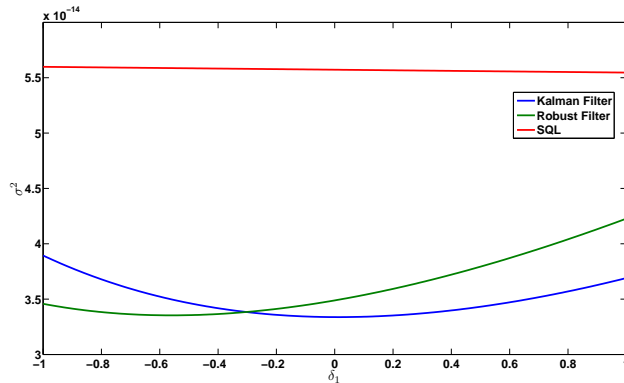


Fig. 5. Comparison of estimation error covariance σ^2 for the different filters with $\mu_1 = 0.2$ and $\mu_2 = 0$.

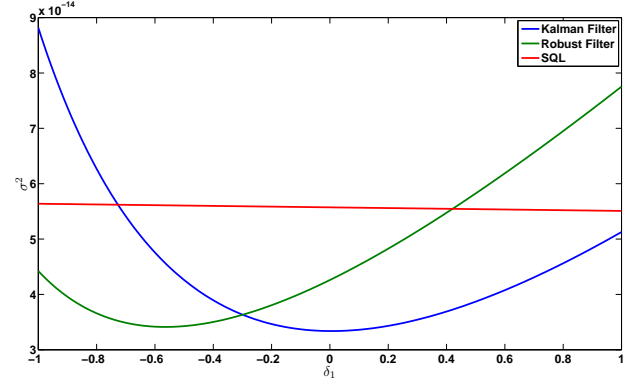


Fig. 6. Comparison of estimation error covariance σ^2 for the different filters with $\mu_1 = 0.5$ and $\mu_2 = 0$.

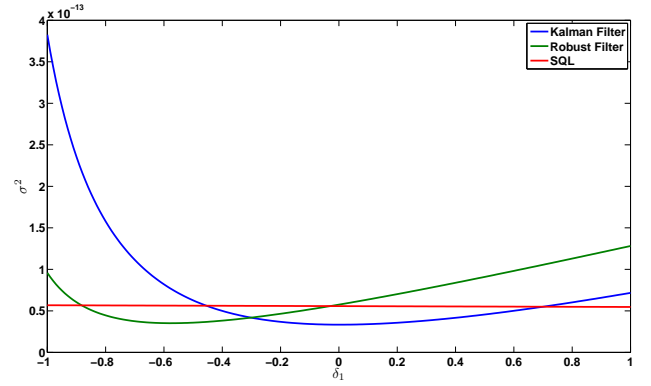


Fig. 7. Comparison of estimation error covariance σ^2 for the different filters with $\mu_1 = 0.8$ and $\mu_2 = 0$.

parameters and chosen values for μ_2 with $\mu_1 = 0$. Figs. 8, 9 and 10 show these plots for $\mu_2 = 0.3, 0.5$ and 0.9 , respectively.

In this case too, the robust filter outperforms the Kalman filter as δ_2 tends towards -1 . The SQL has not been shown in these plots since it is way above these errors.

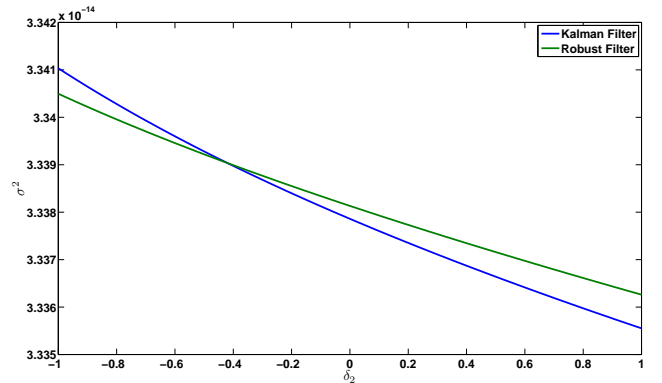


Fig. 8. Comparison of estimation error covariance σ^2 for the different filters with $\mu_1 = 0$ and $\mu_2 = 0.3$.

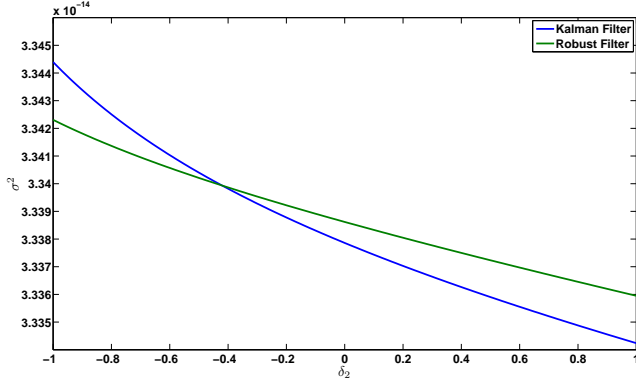


Fig. 9. Comparison of estimation error covariance σ^2 for the different filters with $\mu_1 = 0$ and $\mu_2 = 0.5$.

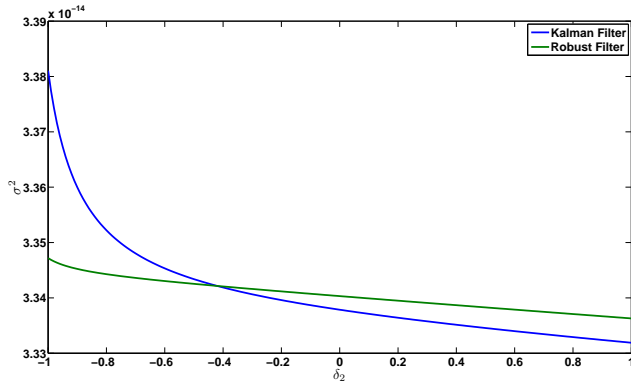


Fig. 10. Comparison of estimation error covariance σ^2 for the different filters with $\mu_1 = 0$ and $\mu_2 = 0.9$.

VI. CONCLUSION

This paper applies guaranteed cost robust filtering theory to the problem of robustly estimating the optical phase of a coherent state evolving continuously as a classical resonant noise process. We showed the behaviour of the robust filter as compared to the Kalman filter with uncertainties in the important parameters underlying the phase noise. The theory herein may be extended to include robust smoothing rather than filtering alone. Robustness to uncertainties in other parameters, such as the photon flux and noise power, may also be explored. Moreover, robustness for a squeezed state, instead of a coherent state, under the influence of such a resonant noise process would be interesting to investigate.

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